Algorithms: UNION FIND AND MINIMUM SPANNING TREES

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EPFL School of Computer and Communication Sciences

Lecture 19, 29.04.2025

Recall: The Ford-Fulkerson Method

Max-flow

Start with 0-flow
while there is an augmenting path from s to t in residual network **do**

- Find augmenting path
- Compute bottleneck= min capacity on path
- Increase flow on the path by the bottleneck

When finished, resulting flow is maximal

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S and T define a minimum cut

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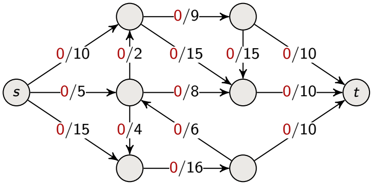

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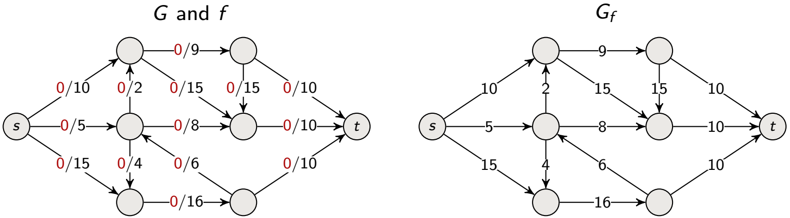

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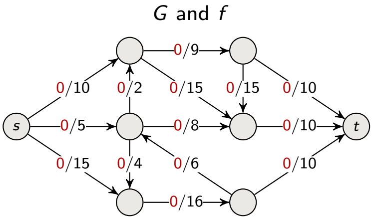

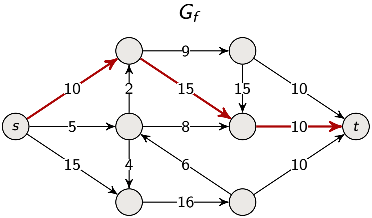

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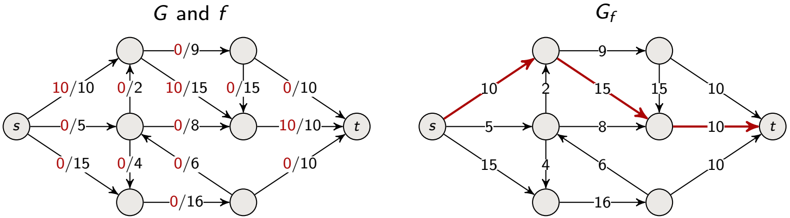

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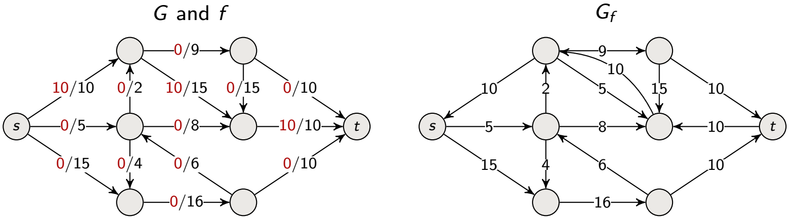

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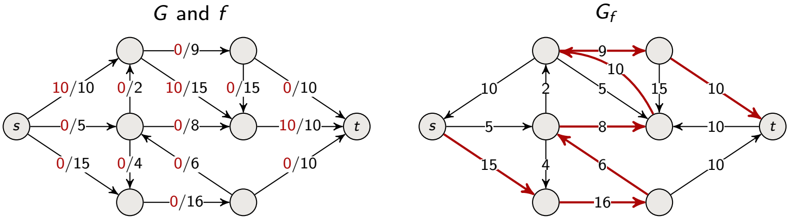

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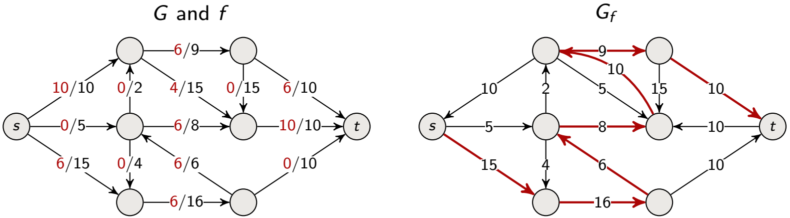

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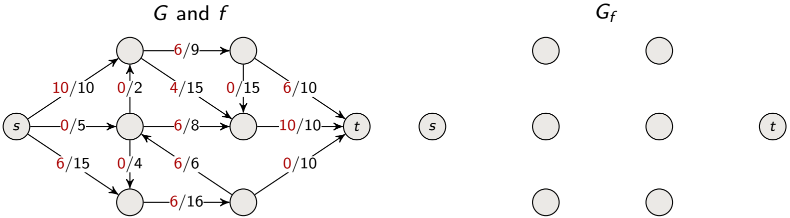

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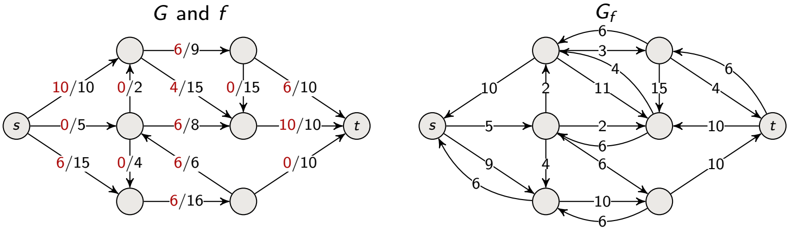

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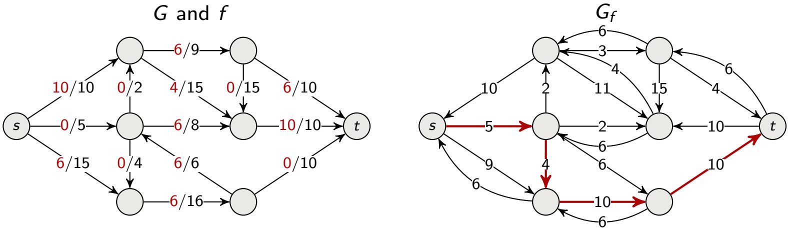

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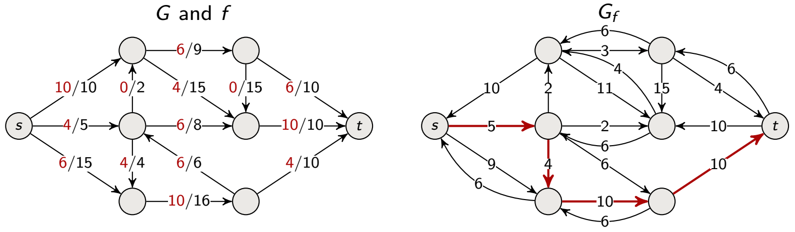

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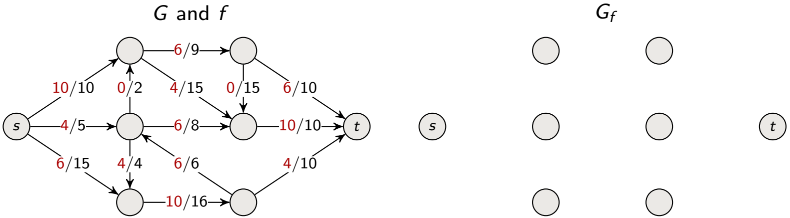

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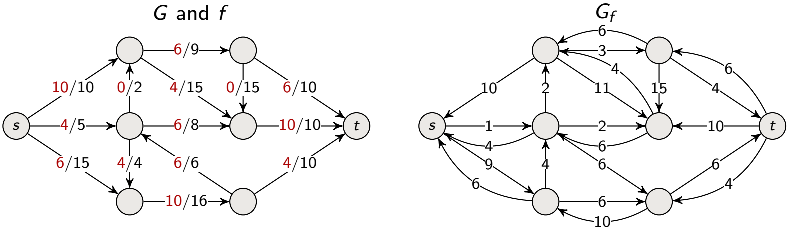

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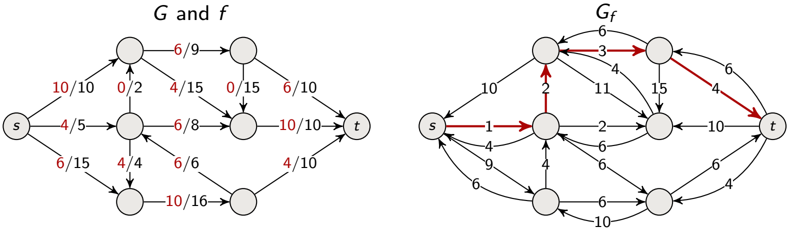

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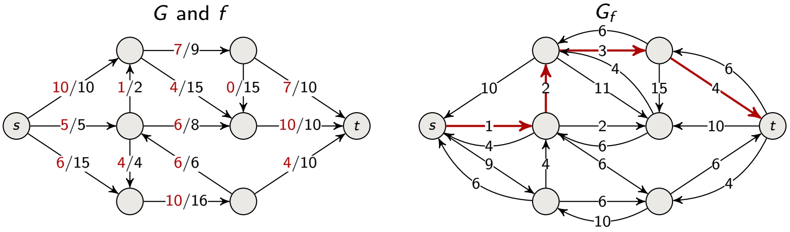

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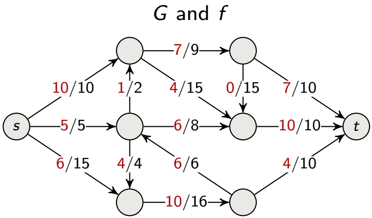

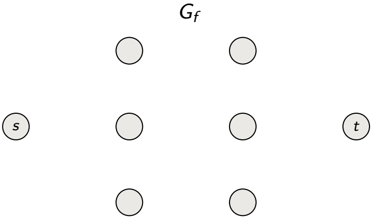

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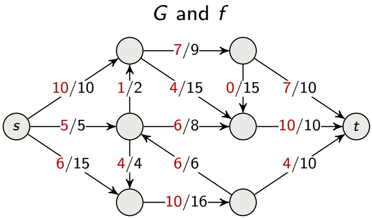

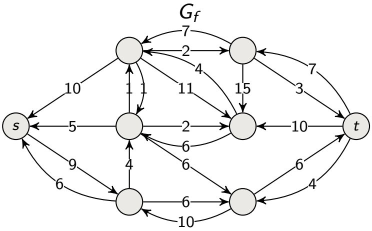

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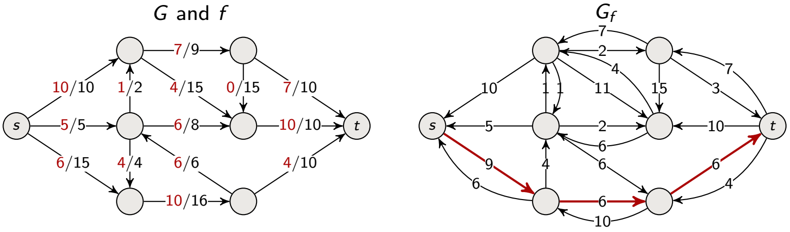

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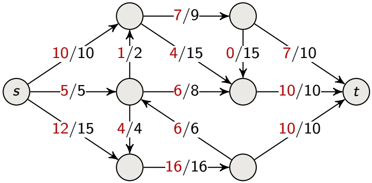

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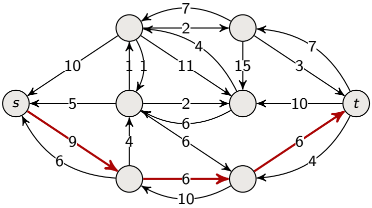

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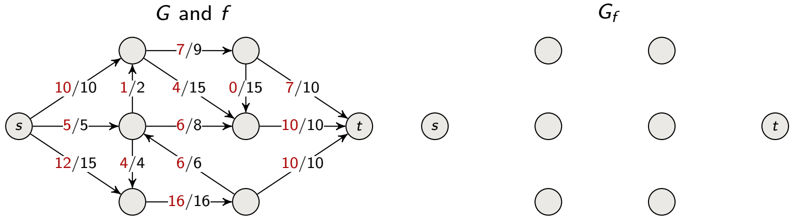

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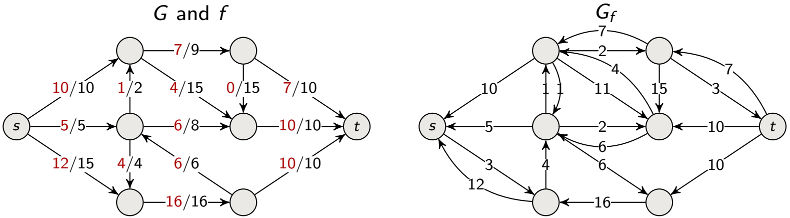

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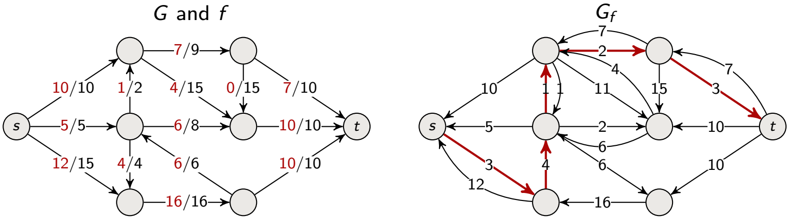

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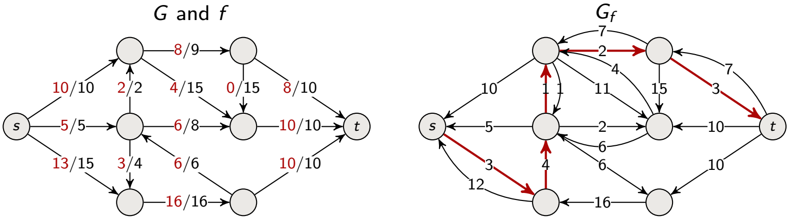

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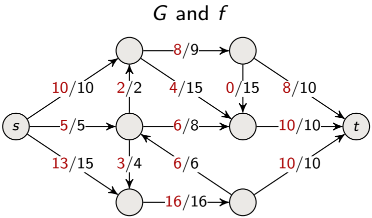

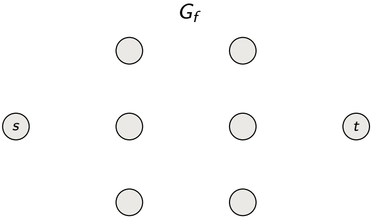

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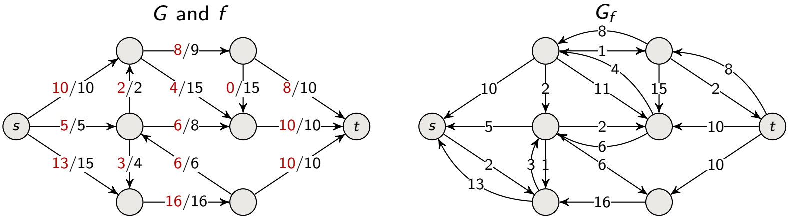

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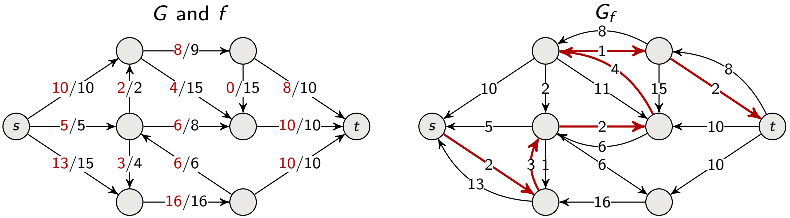

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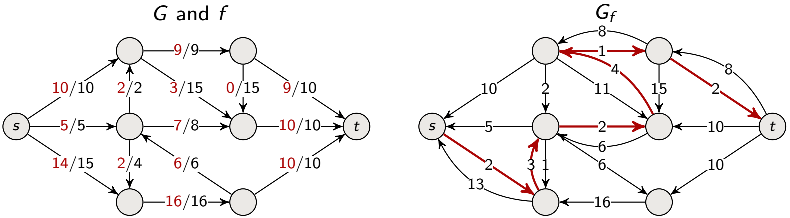

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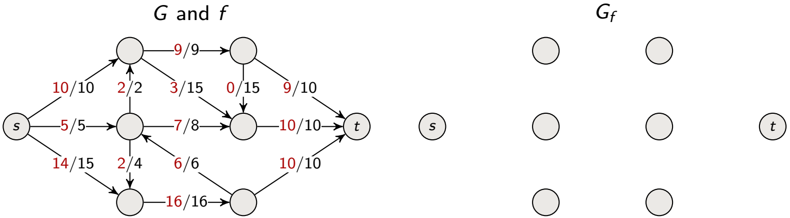

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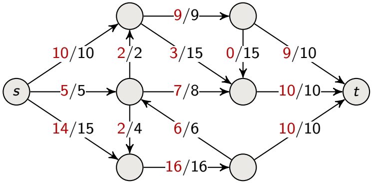

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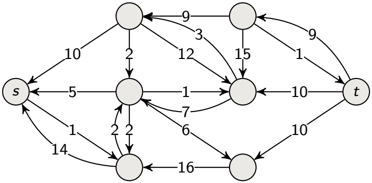

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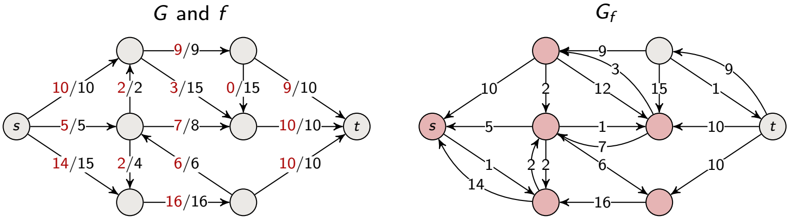

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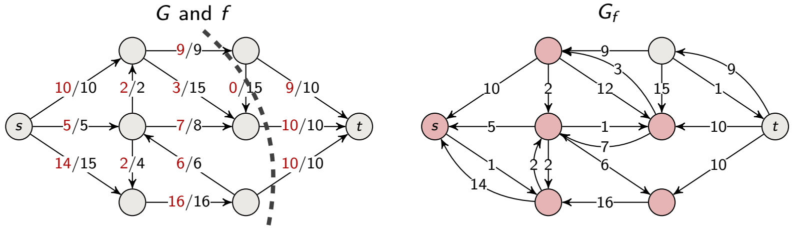

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No augmenting path, flow of value 29 and cut of capacity 29

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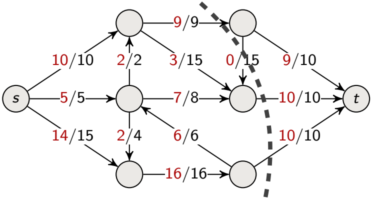

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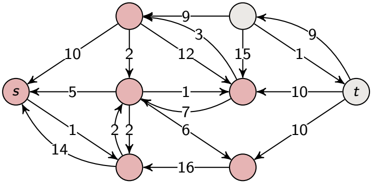

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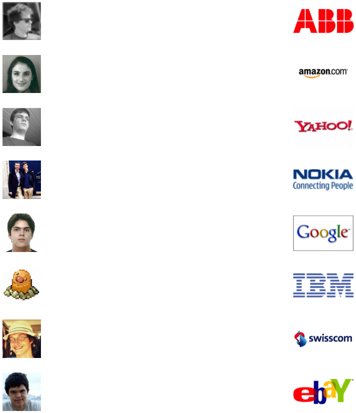
Fattest path	$\leq E \cdot \log(E \cdot U)$

- U is the maximum flow value
- Fattest path: choose augmenting path with largest minimum capacity (bottleneck)

APPLICATIONS OF MAX-FLOW

Bipartite matching

- N students apply for M jobs



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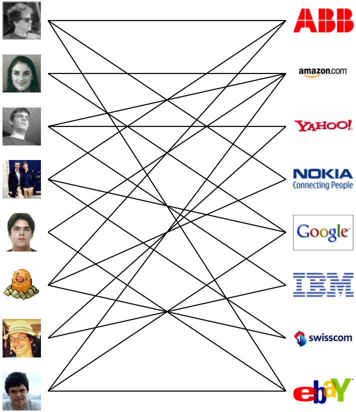
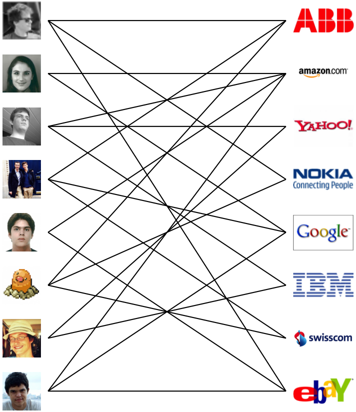
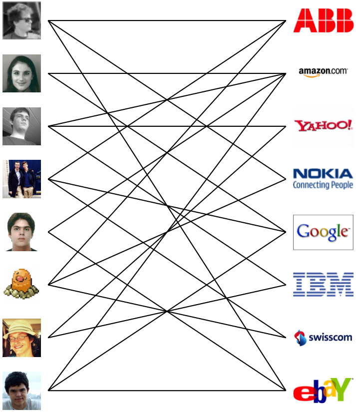


Bipartite matching

- N students apply for M jobs
- Each get several offers
- Is there a way to match all students to jobs? obviously M has to be at least equal to N

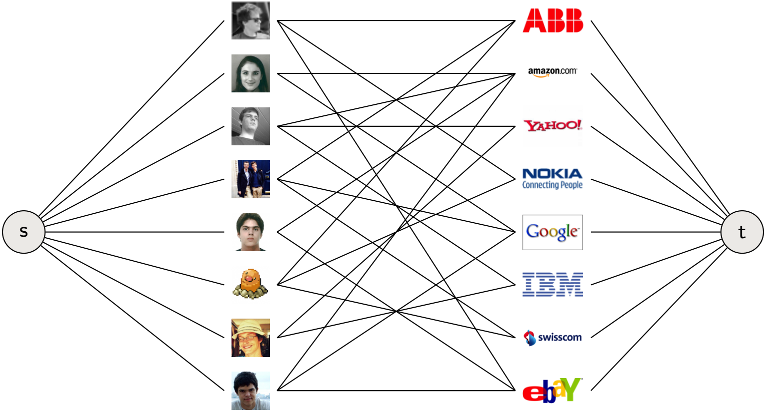


Bipartite matching as flow problem



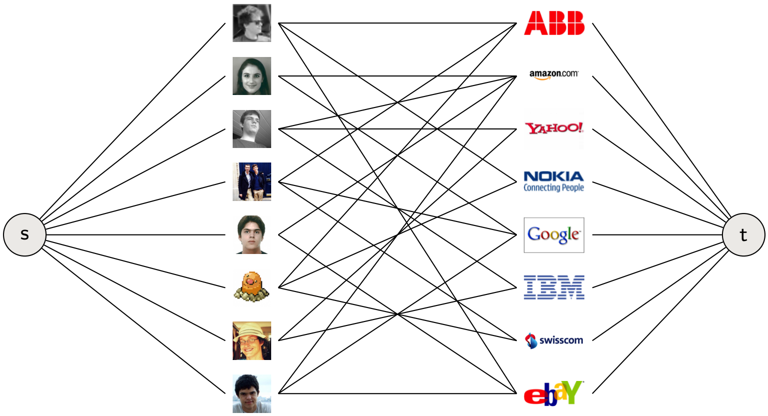
Bipartite matching as flow problem

- Add source s and sink t with edges from s to students and from jobs to t



Bipartite matching as flow problem

- Add source s and sink t with edges from s to students and from jobs to t
- All edges have capacity one



Bipartite matching as flow problem

- Add source s and sink t with edges from s to students and from jobs to t
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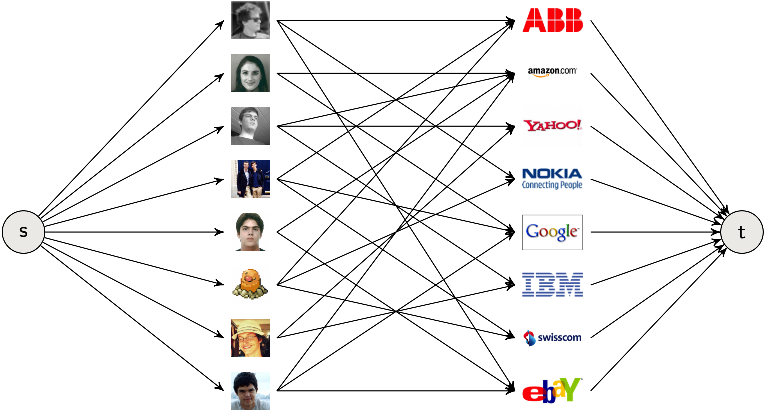
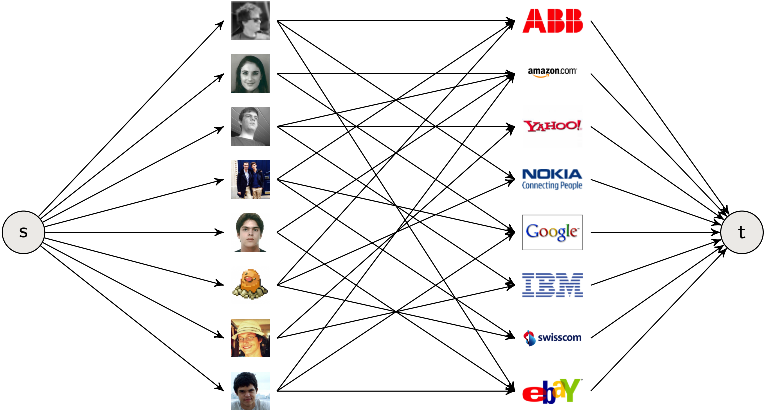
- Direction is from left to right



Bipartite matching as flow problem

- Run the Ford-Fulkerson method



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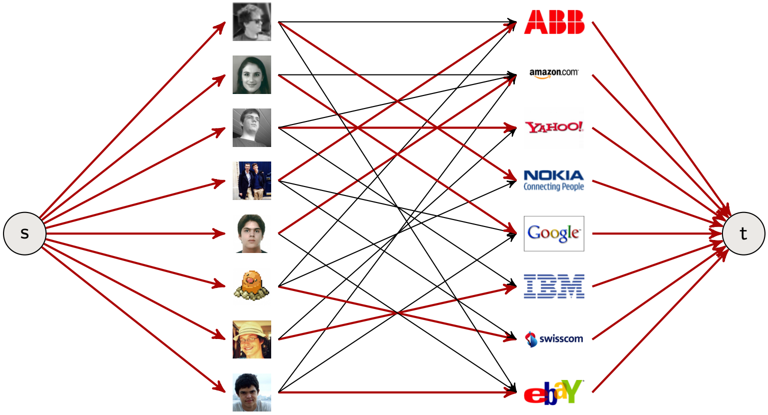

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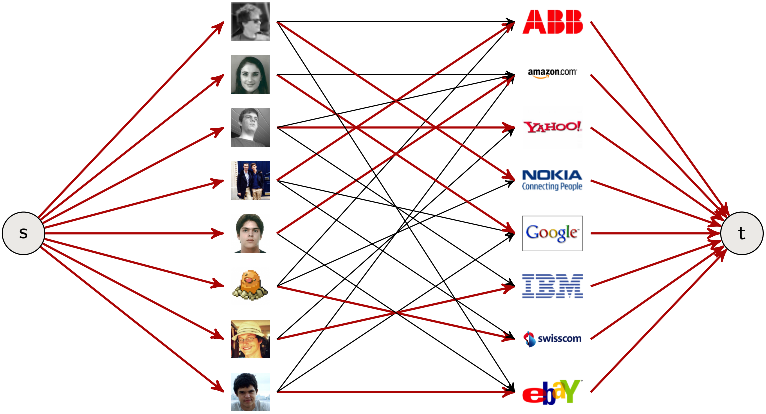
- Run the Ford-Fulkerson method
- Matching is complete



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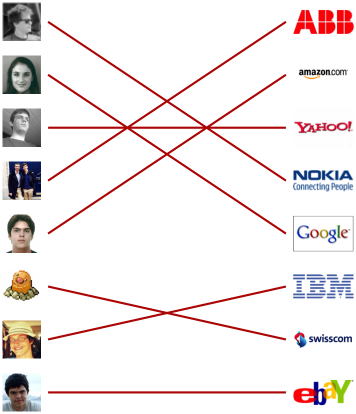

Why does it work?

Every matching defines a flow of value equal to the number of edges in matching

- Put flow 1 on
  - Edges of the matching
  - Edges from s to matched student nodes
  - Edges from matched job nodes to t
- Put flow 0 on all other edges

Works because flow conservation is equivalent to: no student is matched more than once, no job is matched more than once

Why does it work?

Every flow during the algorithm defines a matching of size equal to its value

- Flows obtained by Ford-Fulkerson are integer valued if capacities are integral, so value on every edge is 0 or 1
- Edges between students and jobs with flow 1 are a matching by flow conservation
  - There cannot be more than one edge with flow 1 from a student node
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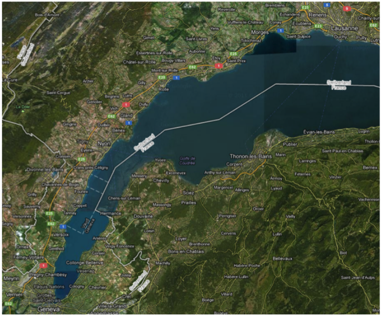
So, maximum flow is a maximum matching!

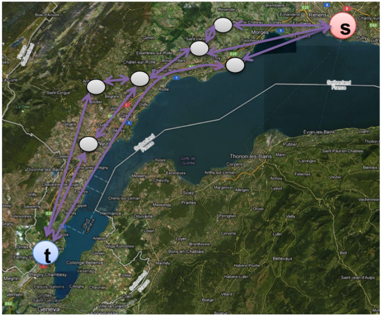
Edge-disjoint paths

- You want to travel to a nice location these winter holidays
- You need to drive from Lausanne to Geneva airport
- Winter season $\Rightarrow$ risk that roads are closed
- How many different routes can you take that does not share a common road?



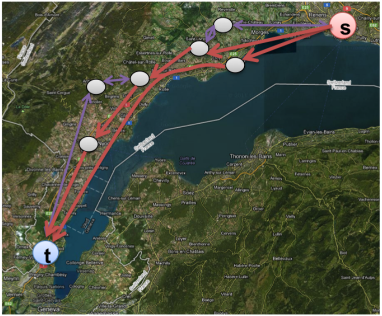
Edge-disjoint paths as flow network

- s = Lausanne
- t = Geneva airport
- An edge capacity of 1 in both directions for each road
- (make anti-parallel using gadgets)



Solution

- max-flow = # edge-disjoint paths
- min-cut = min #roads to be closed so that there is no route from Lausanne to Geneva airport



DATA STRUCTURES FOR DISJOINT SETS

Disjoint-set data structures

- Also known as "union find"
- Maintain collection $\mathcal{S} = \{S_1, \ldots, S_k\}$ of disjoint dynamic (changing over time) sets
- Each set is identified by a representative, which is some member of the set

  Doesn't matter which member is the representative, as long as if we ask for the representative twice without modifying the set, we get the same answer both times

Operations

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MAKE-SET(x): make a new set $S_i = \{x\}$, and add S_i to $\mathcal{S}$

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- Representative of new set is any member in $S_x \cup S_y$, often the representative of one of S_x and S_y
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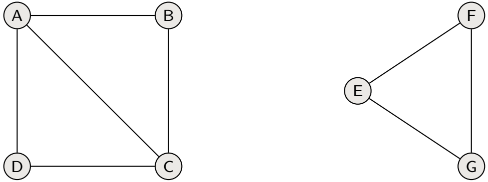
$\textsc{Find}(x)$: return representative of set containing x

Example application: connected components

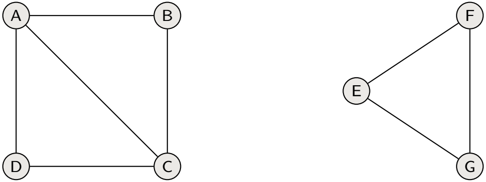
For a graph $G = (V, E)$, vertices u, v are in same connected component if and only if there is a path between them.

- Connected components partition vertices into equivalence classes



Connected components

CONNECTED-COMPONENTS(G)
for each vertex $v \in G.V$
    MAKE-SET(v)
for each edge $(u, v) \in G.E$
    if FIND-SET(u) $\neq$ FIND-SET(v)
        UNION(u, v)



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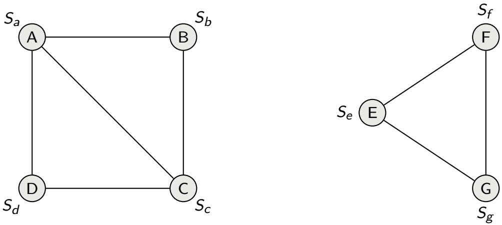

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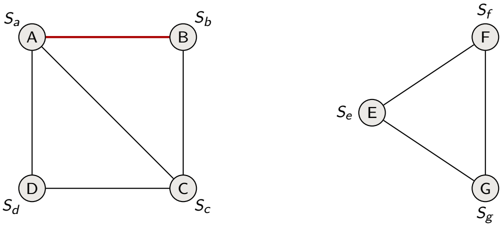

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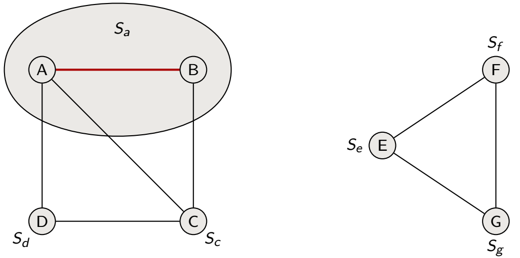

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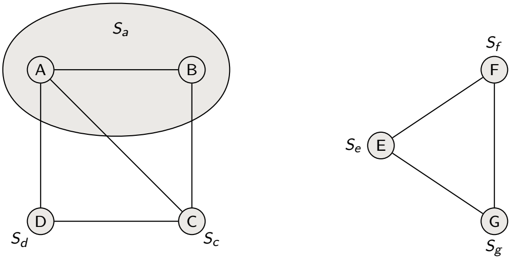

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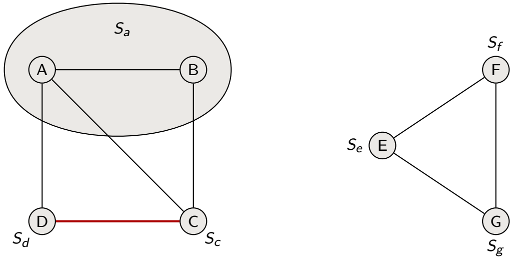

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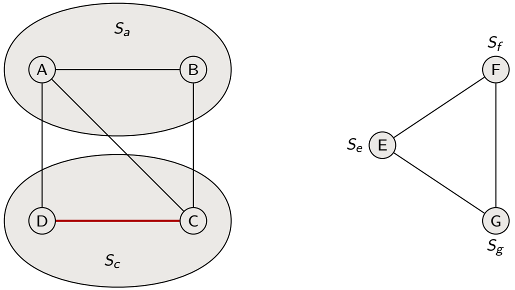

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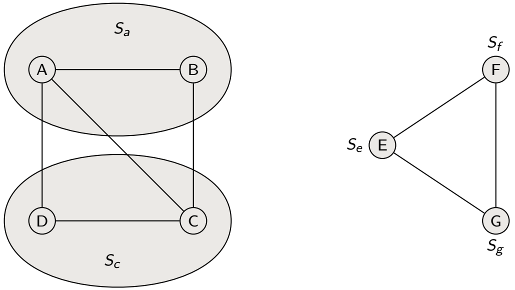

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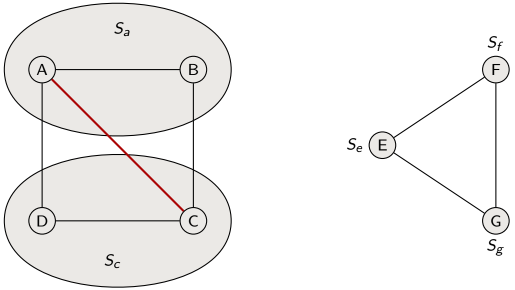

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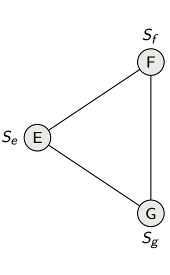

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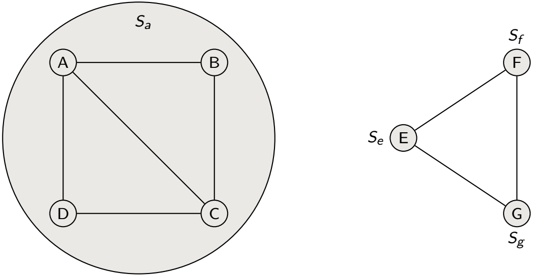

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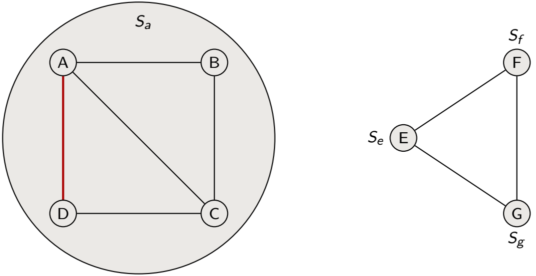

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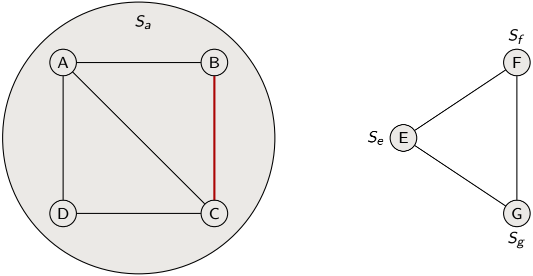

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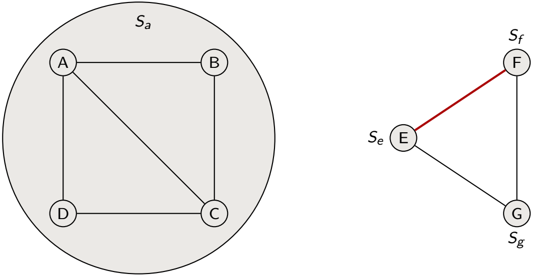

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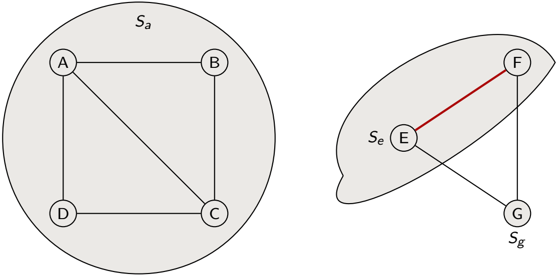

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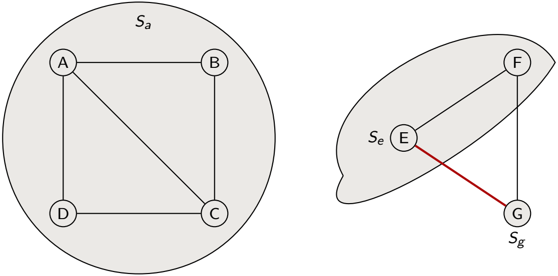

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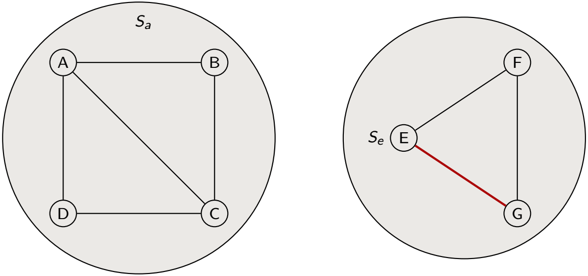

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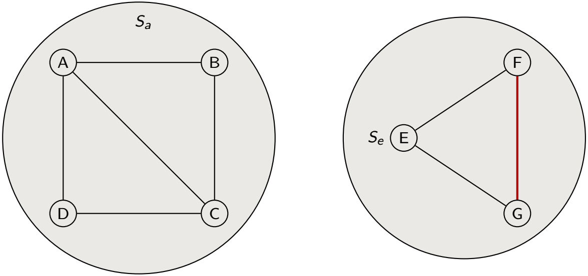

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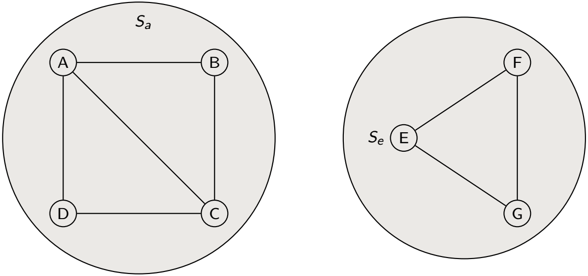
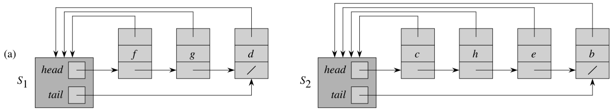


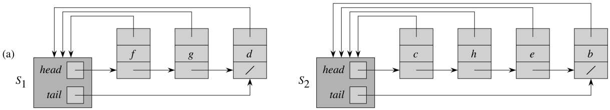
Linked list representation

List representation

- Each set is a single linked list represented by a set object that has
  - a pointer to the *head* of the list (assumed to be the representative
  - a pointer to the *tail* of the list
- Each object in the list has attributes for the *set member*, *pointer to the set object* and *next*



Make-Set and Find



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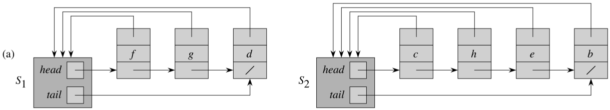
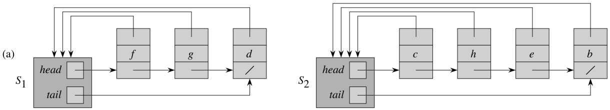


Make-Set and Find

MAKE-SET(x): Create a single ton list in time $\Theta(1)$

FIND(x): follow the pointer back to the list object, and then follow the *head* pointer to the representative (time $\Theta(1)$)



Union

A couple of ways of doing it

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1. Append y's list onto the end of x's list. Use x's tail pointer to find the end.
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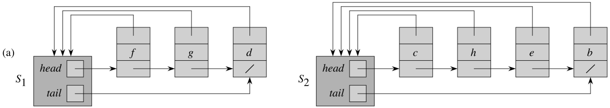

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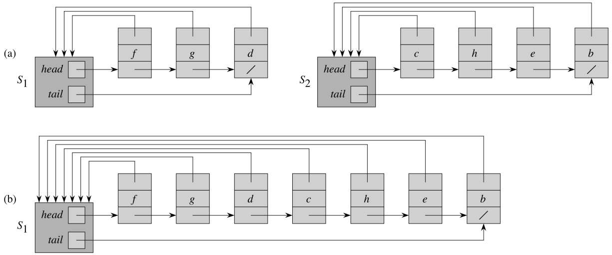

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Operation	Number of objects updated
MAKE-SET(x_1)	1
MAKE-SET(x_2)	1
$\vdots$	$\vdots$
MAKE-SET(x_n)	1
UNION(x_2, x_1)	1
UNION(x_3, x_2)	2
UNION(x_4, x_3)	3
$\vdots$	$\vdots$
UNION(x_n, x_{n-1})	$n-1$

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Proof sketch MAKE-SET and FIND still takes constant time. How many times can each objects' representative pointer be updated? It must be in the smaller set each time

times updated	size of resulting set
1	≥ 2
2	≥ 4
3	≥ 8
$\vdots$	$\vdots$
k	$\geq 2^k$
$\vdots$	$\vdots$
$\log n$	$\geq n$

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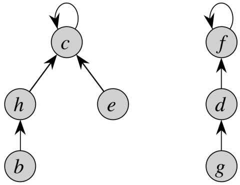
$\log n$	$\geq n$

Therefore, each representative is updated $\leq \log n$ times

Disjoint-set forest

Forest of trees

- One tree per set. Root is representative
- Each node only points to its parent



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Forest of trees

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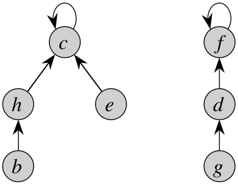
MAKE-SET(x): Make a single-node tree



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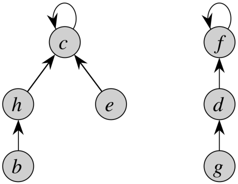

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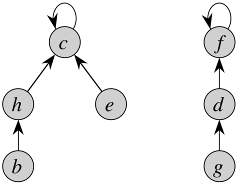
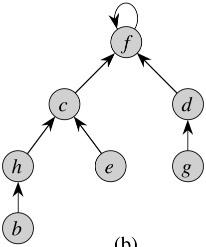
UNION(x, y): make one root a child of another



(a)



(b)

Great heuristics

Union by rank: make the root of the smaller tree a child of the root of the larger tree

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- Use rank, which is an upper bound on height of node
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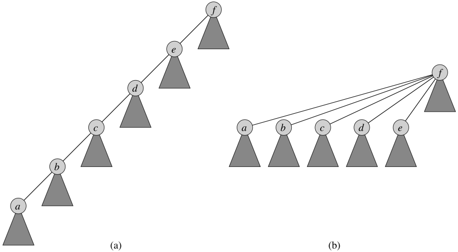
Path compression: **Find path =** nodes visited during FIND on the trip to the root, make all nodes on the find path direct children to root.



Pseudocode of Make-Set and Find-Set

Make-Set(x)

1. $x.p = x$
2. $x.rank = 0$

Pseudocode of Make-Set and Find-Set

```
MAKE-SET(x)
1. x.p = x
2. x.rank = 0
```

```
FIND-SET(x)
1. if x ≠ x.p
2.         x.p = FIND-SET(x.p)
3. return x.p
```

Pseudocode of Union

Union(x, y)

1. Link(Find-Set(x), Find-Set(y))

Pseudocode of Union

```
UNION(x, y)

LINK(FIND-SET(x), FIND-SET(y))
```

```
LINK(x, y)

if x.rank > y.rank
    y.p = x
else x.p = y
    if x.rank == y.rank
        y.rank = y.rank + 1
```

Running time

If use both union by rank and path compression,

$$O(m \cdot \alpha(n))$$

where $\alpha(n)$ is an extremely slowly growing function:

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$4 - 7$	2

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- $\alpha(n) \leq 5$ for any practical purpose
- The bound $O(m \cdot \alpha(n))$ is tight

Running time of connected components

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CONNECTED-COMPONENTS(G)
  for each vertex v ∈ G.V
      MAKE-SET(v)
  for each edge (u, v) ∈ G.E
      if FIND-SET(u) ≠ FIND-SET(v)
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$$O((V + E)\alpha(V)) \approx O(V + E)$$

MINIMUM SPANNING TREES

Origin of today's lecture

Otakar Boruvka (1926)

- Electrical power company in western Moravia in Brno
- Most economical construction of electrical power network
- Concrete engineering problem led to what is now a cornerstone problem-solving model in combinatorial optimization

A spanning tree of a graph

A set **T** of edges that is

- Acyclic
- Spanning (connects all vertices)

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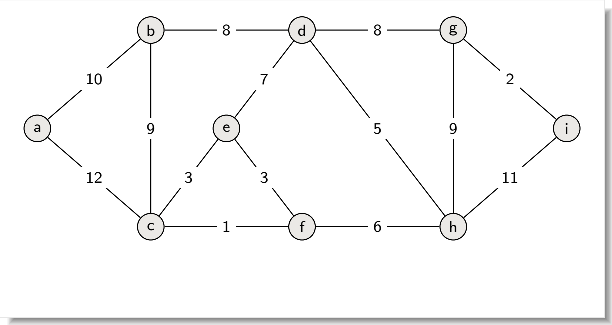

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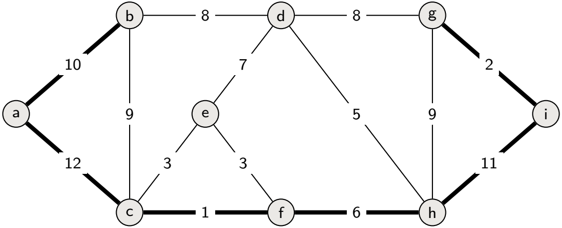

Acyclic, but not connected

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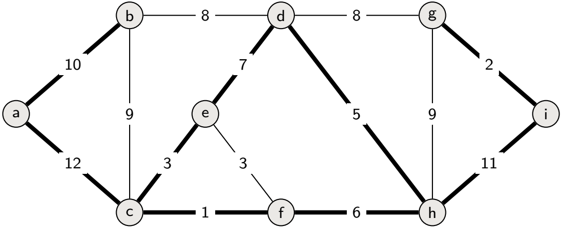

Connected, but not acyclic

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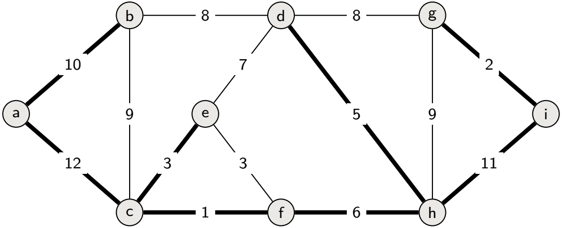
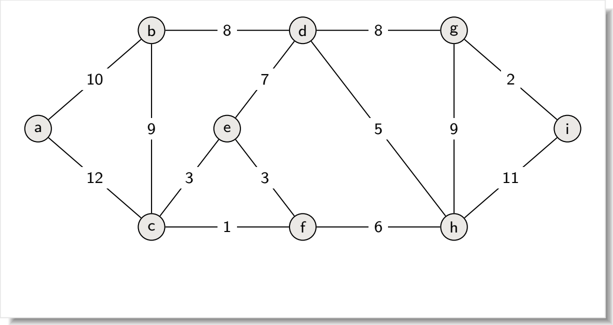


Acyclic and connected = spanning tree

Minimum spanning tree (MST)

INPUT: an undirected graph $G = (V, E)$ with weight $w(u, v)$ for each edge $(u, v) \in E$

OUTPUT: a spanning tree of minimum total weight



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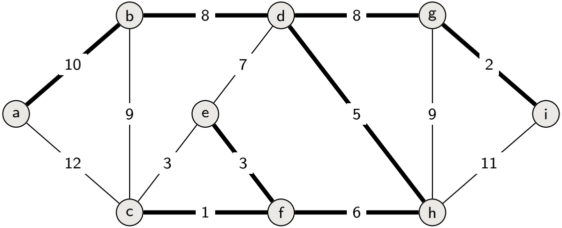

Spanning tree of weight $10 + 8 + 1 + 3 + 8 + 5 + 6 + 2 = 43$

EXAMPLE APPLICATIONS

Example 1: Communication networks

A multinational company wants to lease communication lines between its various locations

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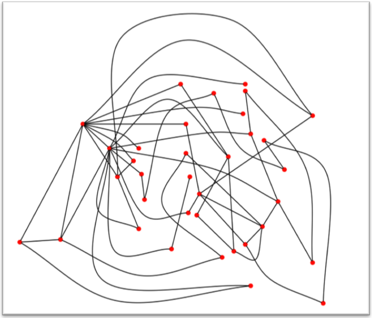

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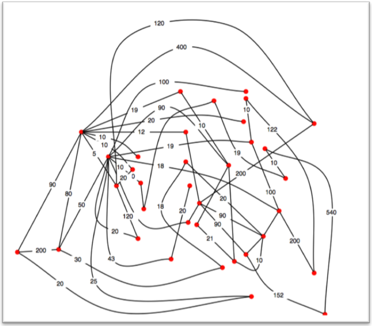
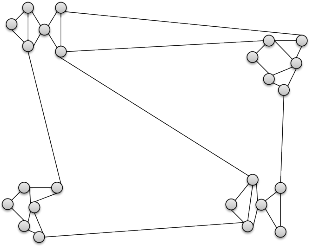


A multinational company wants to lease communication lines between its various locations

Solution given by a MST on the graph

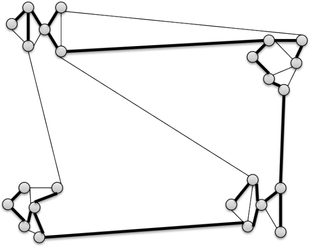
Example 2: Clustering



Edge weights equal to distance of nodes

Find: "cluster" of nodes   Possible solution: Find MST. Eliminate "fat" edges

Example 2: Clustering



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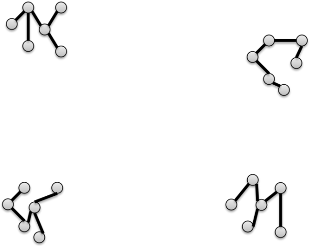

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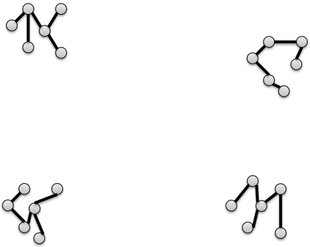
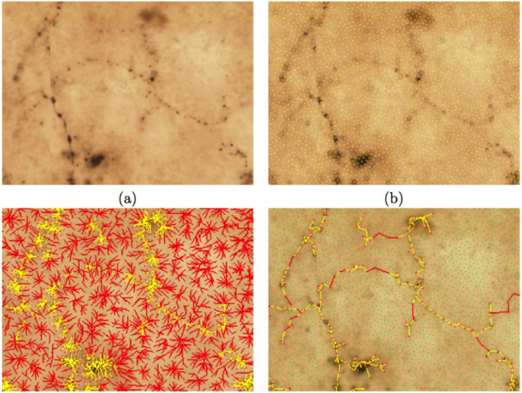


Edge weights equal to
distance of nodes

Find: "cluster" of nodes   Possible solution: Find MST. Eliminate "fat" edges

Note: this is a "heuristic" algorithm. Needs analysis

Example 3: Dendritic structures in the brain



(a) (b) (c) (d)

Example 4: Phylogenetic trees

Infer evolutionary relationships among various biological species

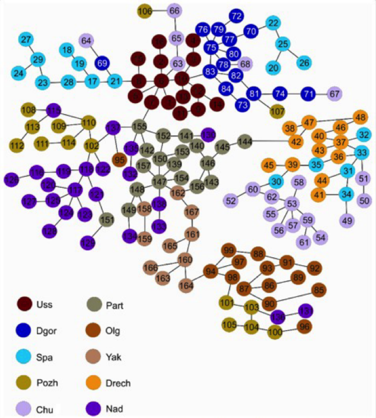
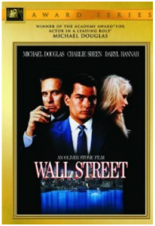


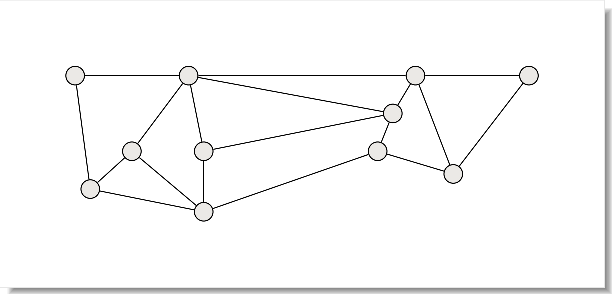
Figure 2 MS-tree representing phylogenetic relationships among representative *Panax ginseng* populations. Length of lines is proportional to the Euclidean distances among plants. Length of scale line is equal to 50 units of Euclidean distances



ALGORITHMS FOR MST

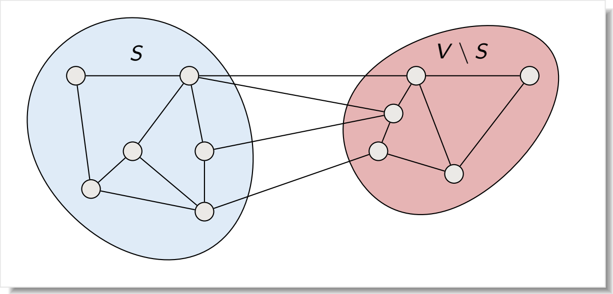
"Greed is good. Greed is right. Greed works. Greed clarifies, cuts through and captures the essence of the evolutionary spirit."
- Gordon Gecko

Cuts



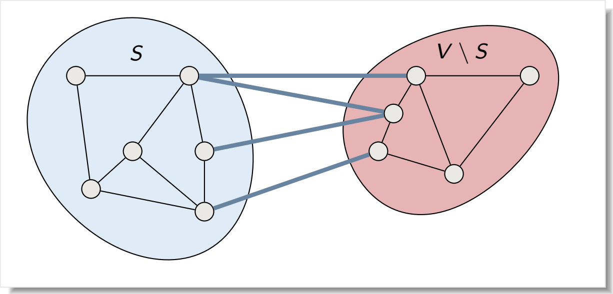
Cuts

- A **cut** $(S, V \setminus S)$ is a partition of the vertices into two nonempty disjoint sets S and $V \setminus S$



Cuts

- A **cut** $(S, V \setminus S)$ is a partition of the vertices into two nonempty disjoint sets S and $V \setminus S$
- A **crossing edge** is an edge connecting vertex S to vertex in $V \setminus S$

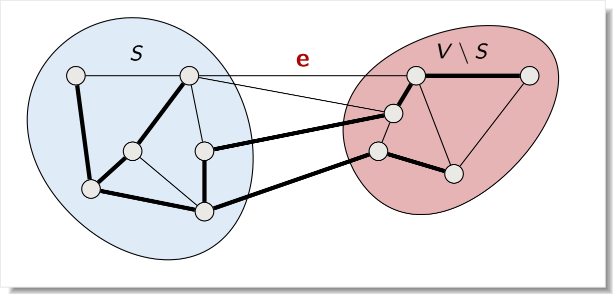


Cut property

Consider a cut $(S, V \setminus S)$ and let

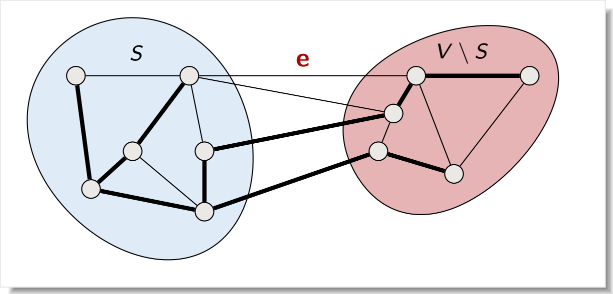
- T be a tree on S which is part of a MST
- e be a crossing edge of minimum weight

Then there is MST of G containing e and T



Cut property

Proof. If e is already in MST we are done.



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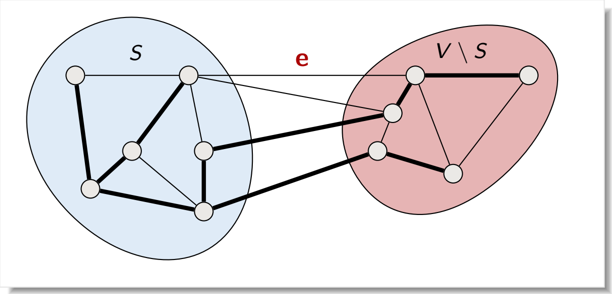
Proof. If e is already in MST we are done.
Otherwise add e to the MST



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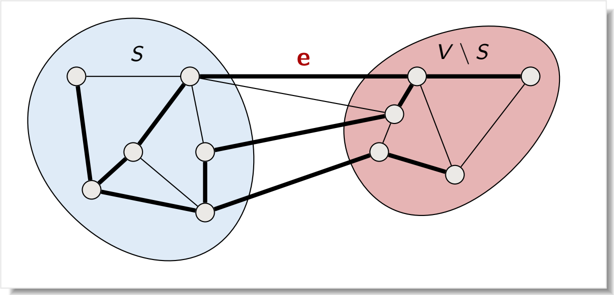

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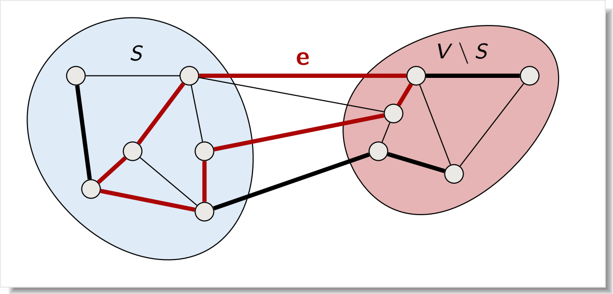
Proof. If e is already in MST we are done.
Otherwise add e to the MST
This creates a cycle



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Proof. If e is already in MST we are done.
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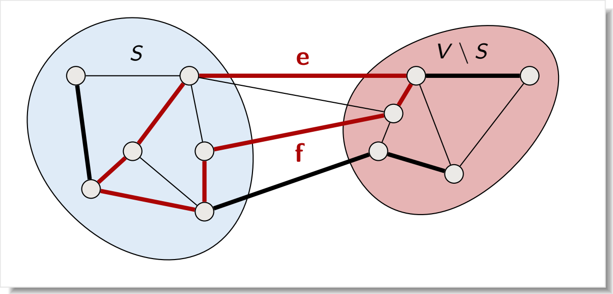
At least one other crossing edge f in cycle



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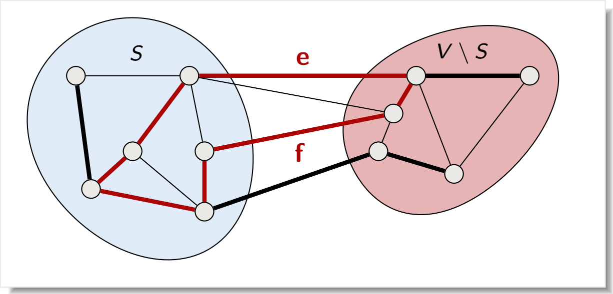

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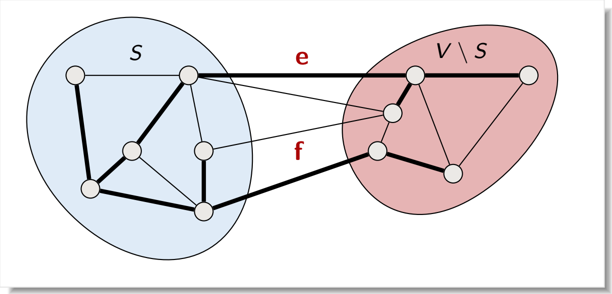
Replace f by e in MST
This gives new MST which contains T and e



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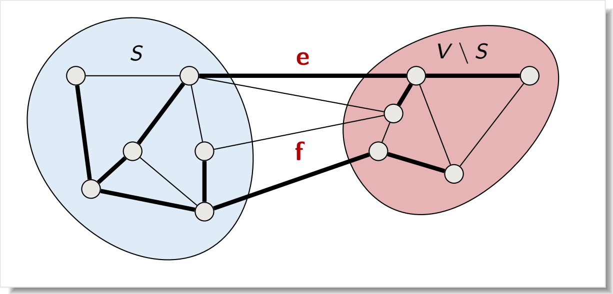
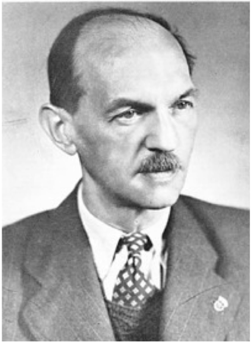


Prim's algorithm



Vojitech Jarnik
1897 - 1970

Robert Prim
1921-

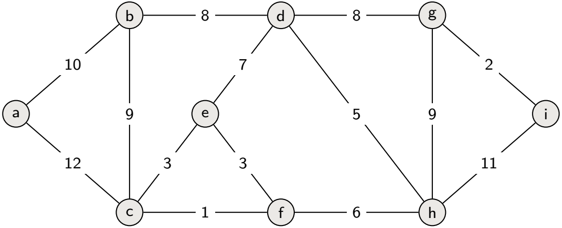
Edsger Dijkstra
1930 - 2002

Prim's algorithm

Start with any vertex v, set tree T to singleton v

Greedily grow tree T:

at each step add to T a minimum weight crossing edge with respect to the cut induced by T



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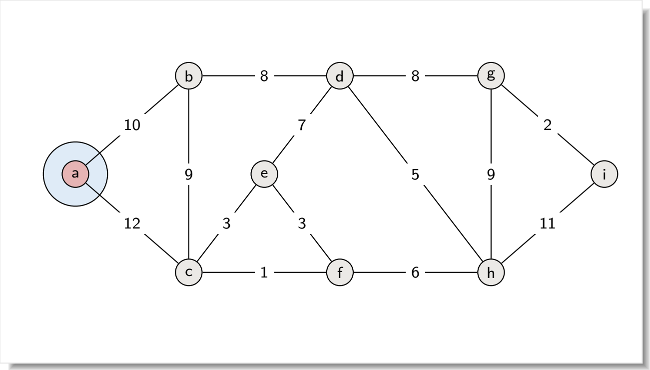

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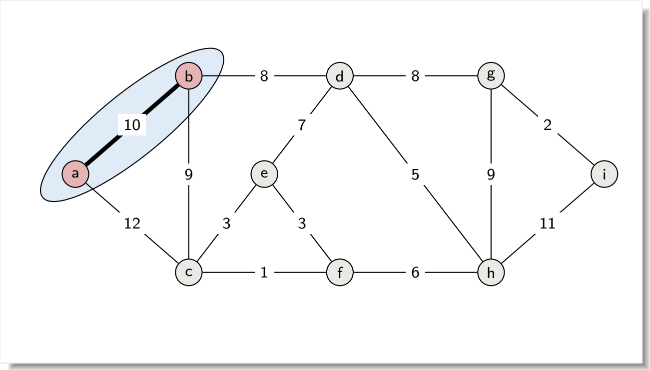

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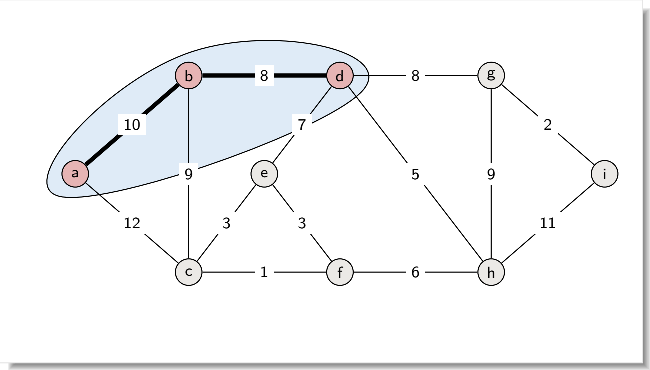

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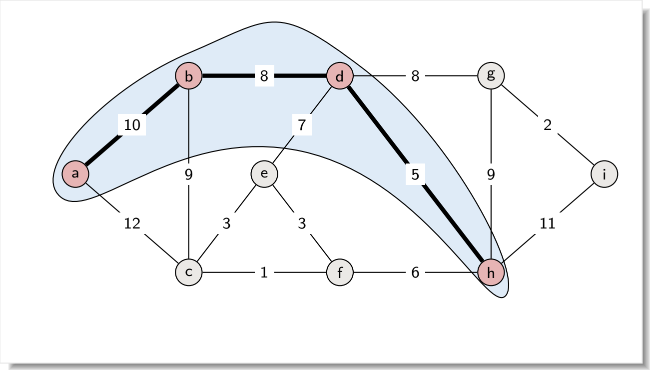

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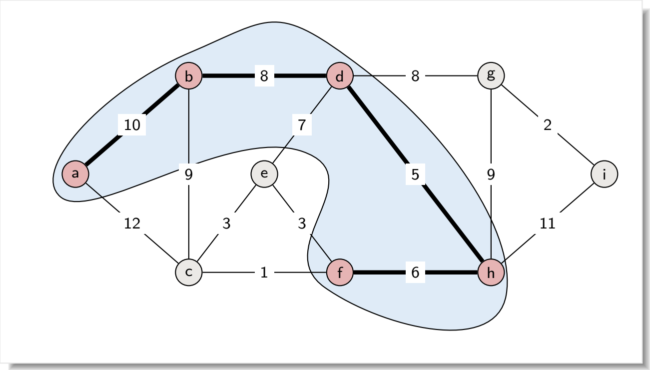

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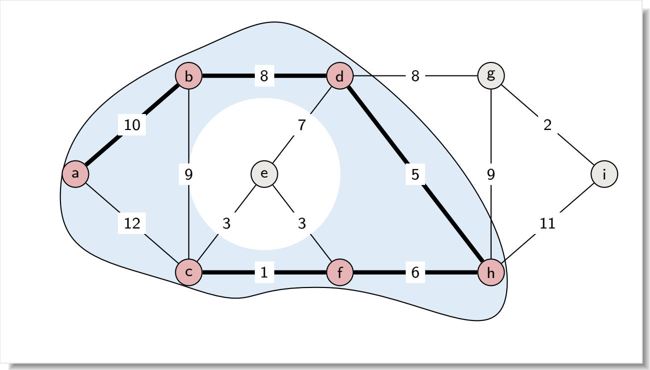

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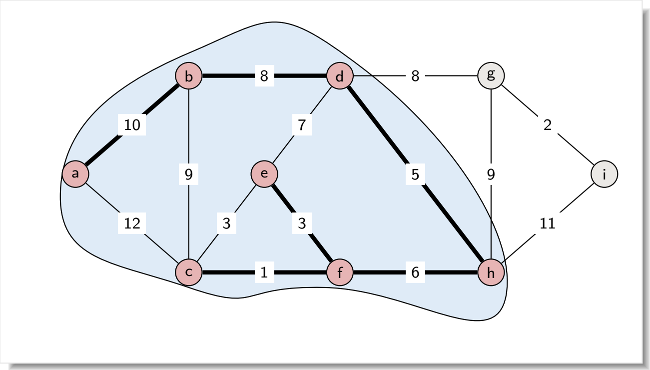

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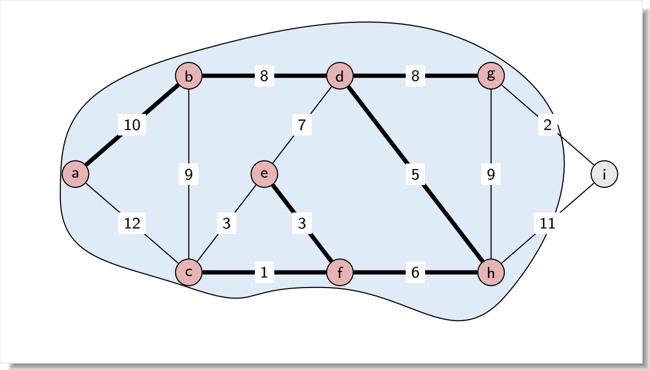

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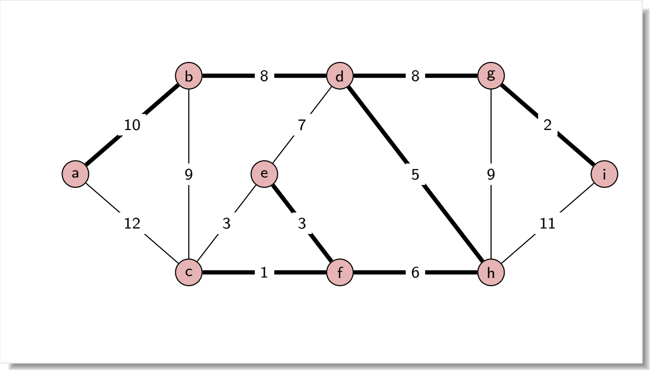

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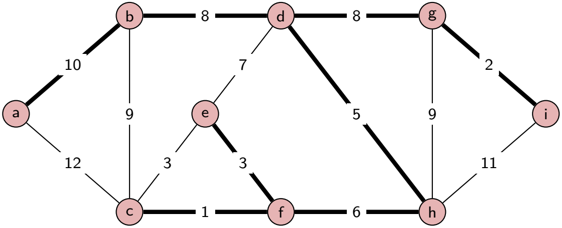

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T is always a subtree of a MST

Proof by induction on number of nodes in T. Final T is MST by this result

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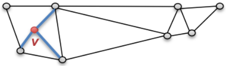

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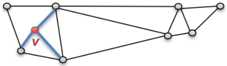
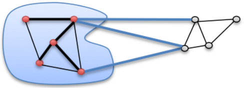
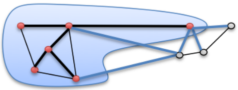


Inductive step: use cut property



In MST by hypothesis



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More clever solution:

- For every node w, keep value $dist(w)$ that measures the "distance" of w from current tree
- When a new node u is added to tree, check whether neighbors of u decreases their distance to tree; if so, decrease distance
- Maintain a min-priority queue for the nodes and their distances

Implementation and Analysis

```
PRIM(G, w, r)
  Q = ∅
  for each u ∈ G.V
      u.key = ∞
      u.π = NIL
      INSERT(Q, u)
  DECREASE-KEY(Q, r, 0)        // r.key = 0
  while Q ≠ ∅
      u = EXTRACT-MIN(Q)
      for each v ∈ G.Adj[u]
          if v ∈ Q and w(u, v) < v.key
              v.π = u
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- Total: $O(E \lg V)$ (can be made $O(E + V \lg V)$ with careful queue implementation)

Summary

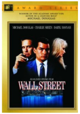
- Greedy is good (sometimes)
- Prim's algorithm

  Min-priority queue for implementation

- Next time Kruskal's algortihm

  Union-Find for implementation

- Many applications



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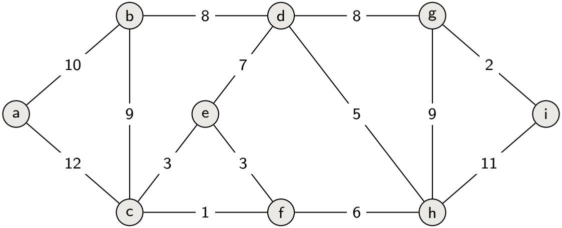
at each step add cheapest edge that does not create a cycle



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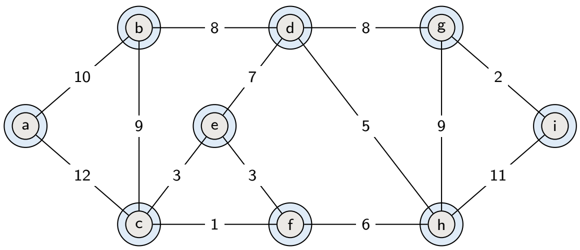

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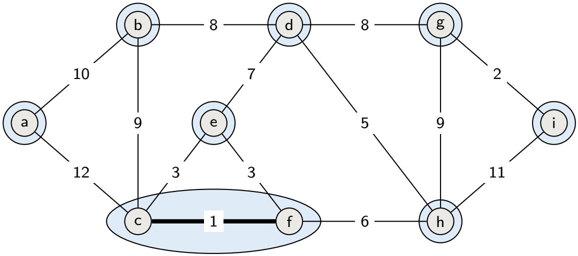

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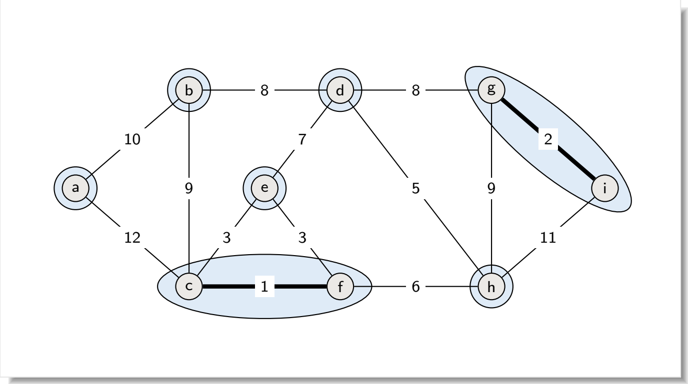

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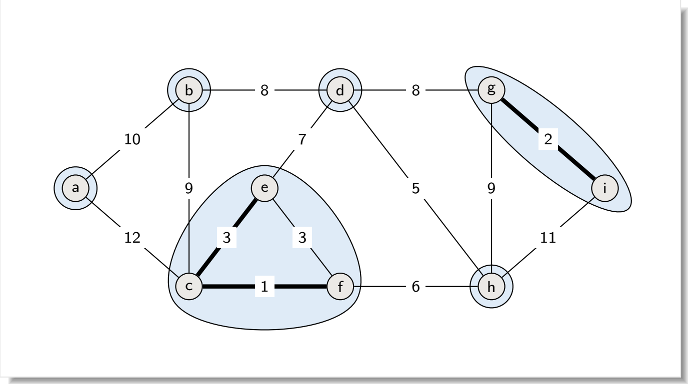

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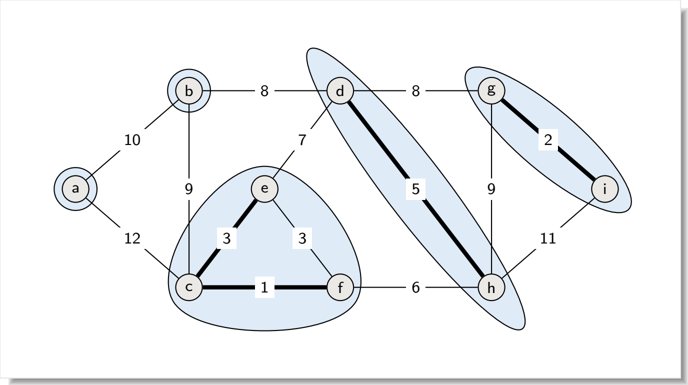

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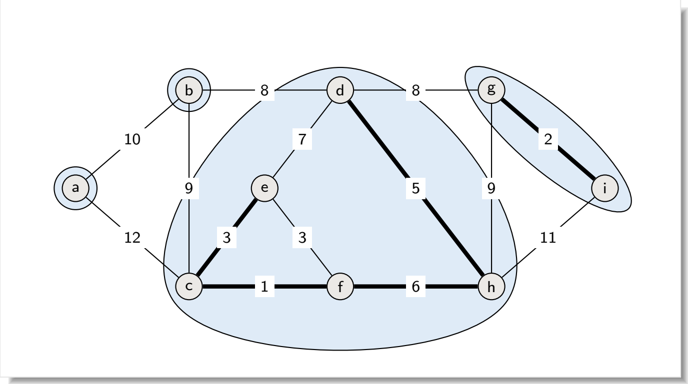

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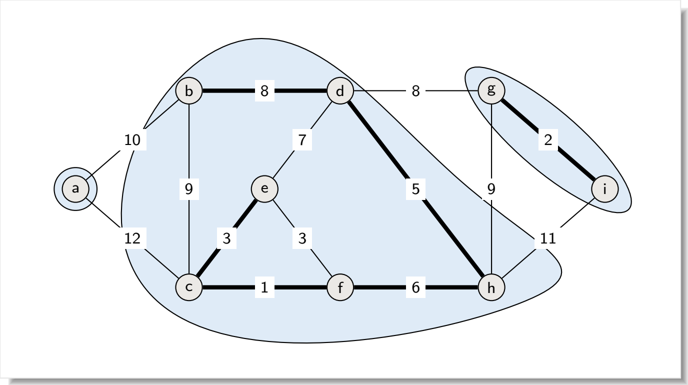

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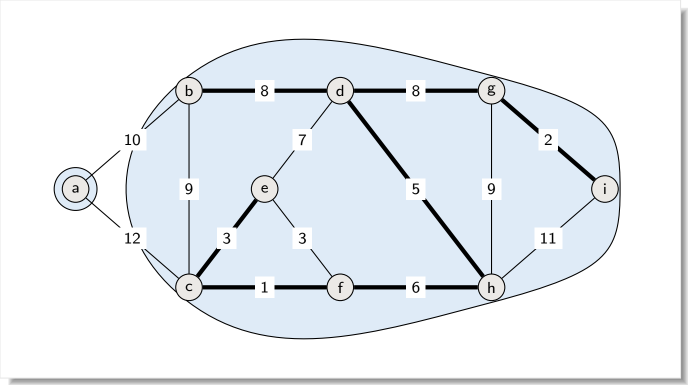

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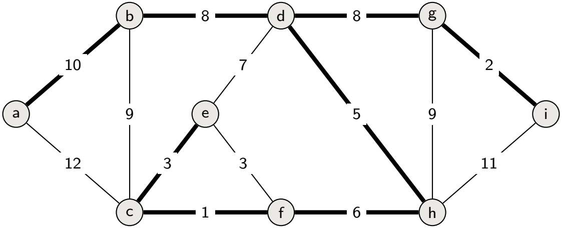

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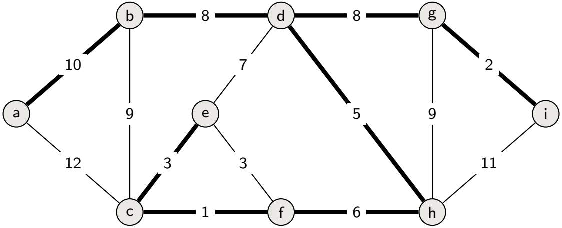

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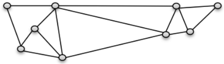

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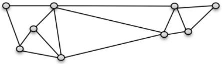

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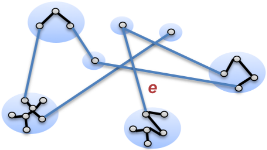

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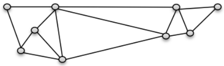

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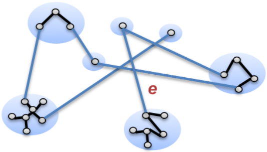

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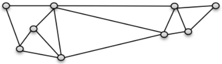

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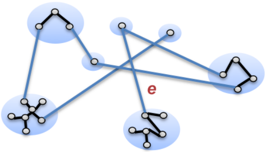

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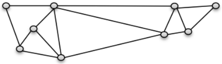

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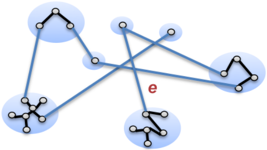
4. Replace an edge (with larger weight) along this cycle by ***e***



An MST since weight did not increase!

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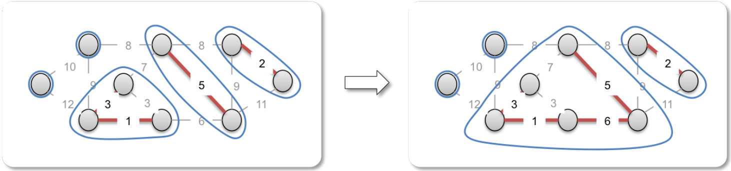

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  sort the edges of $G.E$ into nondecreasing order by weight w
  for each (u, v) taken from the sorted list
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      $A = A \cup \{(u, v)\}$
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```

- Initialize A: $O(1)$
- First **for** loop: V MAKE-SETS
- Sort E : $O(E \lg E)$
- Second **for** loop: $O(E)$ FIND-SETS and UNIONS
- Total time: $O((V + E)\alpha(V)) + O(E \lg E) = O(E \lg E) = O(E \lg V)$

Implementation and Analysis

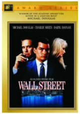
```
KRUSKAL(G, w)
  A = ∅
  for each vertex v ∈ G.V
      MAKE-SET(v)
  sort the edges of G.E into nondecreasing order by weight w
  for each (u, v) taken from the sorted list
      if FIND-SET(u) ≠ FIND-SET(v)
          A = A ∪ {(u, v)}
          UNION(u, v)
  return A
```

- Initialize A: $O(1)$
- First **for** loop: V MAKE-SETS
- Sort E : $O(E \lg E)$
- Second **for** loop: $O(E)$ FIND-SETS and UNIONS
- Total time: $O((V+E)\alpha(V)) + O(E \lg E) = O(E \lg E) = O(E \lg V)$

  If edges already sorted time is $O(E\alpha(V))$ which is almost linear

Summary

- Greedy is good (sometimes)



- Prim's algorithm

  Min-priority queue for implementation

- Kruskal's algortihm

  Union-Find for implementation

- Many applications